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On Properties of Generalized Quaternion Algebra

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ABSTRACT: In this paper, we study the generalized quaternions, $H_{\alpha\beta}$, and their algebraic properties. De Moivre's and Euler's formulas for these quaternions in different cases are investigated. The solutions of equation $q^n = 1$ is discussed mean while it has been shown that equation $q^n = 1$has uncountably many solutions for unit generalized quaternions. Finally, the relations between the powers of these quaternions are given.

Keywords: De Moivre’s formula, Generalized Quaternion, Lie group.

INTRODUCTION

Mathematically, quaternions represent the natural extension of complex numbers, forming an associative algebra under the addition and the multiplication. This algebra is an effective way for understanding many aspects of physics and kinematics. Nowadays, quaternions are used especially in the area of computer vision, computer graphics, animation, and to solve optimization problems involving the estimation of rigid body transformations as well. Obtaining the roots of a quaternion was given by Niven, (1942) and Brand, (1942). Brand proved De Moivre’s theorem and used it to find nth roots of a quaternion. Using De Moivre’s formula to find roots of a quaternion is more convenient. These formulas are also investigated in the cases of dual, split and complex quaternions (Kabadayi and Yayli, 2011; Ozdemir, 2009; Jafari, 2013). Whittlesey and Whittlesey, (1990) by the help of Euler's formula found the circles in the plane and the sphere in 3-space by means of the exponential expansions. In this paper, we briefly recall some fundamental properties of the generalized quaternions, and show that the set of all unit generalized quaternions with the group operation of quaternion multiplication is a Lie group of 3-dimension. Moreover, we obtain De-Moivre's and Euler's formulas for these quaternions in different cases. We use it to find n-th roots of a generalized quaternion. Finally, we give some example for the purpose of more clarification.

Preliminaries

The Irish mathematician Rowan Hamilton struggled in vain to extend complex numbers to three dimensions. Eventually, he realized that it is necessary to go to four dimensions and he invented a new number system called the quaternions. Although, Hamilton did not use the ordered pair construction for quaternions, but he was the inventor of the pair construction for complex numbers.

A quaternion, is an ordered pair of complex numbers z_1, z_2 *i.e.*

$q=(z_1,z_2),\ z_1$ and $z_2 \in \mathbb{C}$,

with addition and multiplication defined by

$$(z_1,z_2)+(w_1,w_2)=(z_1+w_1,z_2+w_2),$$

$$(z_1,z_2)(w_1,w_2)=(z_1w_1-z_2w_2^*,z_1w_2+z_2w_1^*),$$

and

$$a(z,w)=(a,0)(z,w)=(az,aw), \qquad a\in\mathbb{R}.$$

It turns out that multiplication is not commutative. That is, in general for quaternions q, r we have

$rq \neq qr.$

This construction by pairs ties in nicely with the constructions of the rational, real, and complex numbers but is not the traditional approach. If we single out three special pairs and attach Hamilton's notation to them as

$$\vec{i} = (\iota, 0), \;\; \vec{j} = (0,1), \;\; \vec{k} = (0, \iota)$$

and identify

$$(a,0) \leftrightarrow \mathrm{a} \qquad a \in □,$$

then we find

$$(a_0 + \iota a_1, a_2 + \iota a_3) = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \qquad a_i \in □$$

which is the form Hamilton used to express quaternions. This form makes it quite clear that quaternions are a four-dimensional generalization of complex numbers.

The quaternions $\vec{i}, \vec{j}, \vec{k}$ satisfy the following relations:

$$\vec{i}^{\,2} = \vec{j}^{\,2} = \vec{k}^{\,2} = \vec{i}\,\vec{j}\,\vec{k} = -1.$$

In the language of abstract algebra, the quaternions form a noncommutative, normed division algebra over □. The eight-dimensional octonians O can be constructed from pairs of quaternions but there the chain ends. The only normed division algebras over □ are □, □, H and O. (Heard, 2006).

Generalized Quaternion Algebra

This section summarizes the essentials of the algebra of generalized quaternions. A generalized quaternion q is an expression of the form

$$q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

where a_0, a_1, a_2 and a_3 are real numbers and $\vec{i}, \vec{j}, \vec{k}$ are quaternionic units satisfying the equalities

$$\vec{i}^{\,2} = -\alpha, \;\; \vec{j}^{\,2} = -\beta, \;\; \vec{k}^{\,2} = -\alpha\beta,$$

$$\vec{i}\vec{j} = \vec{k} = -\vec{j}\vec{i}, \;\; \vec{j}\vec{k} = \beta\vec{i} = -\vec{k}\vec{j},$$

and

$$\vec{k}\vec{i} = \alpha\vec{j} = -\vec{i}\vec{k}, \quad \alpha, \beta\ \hat{I}\ R.$$

The set of all generalized quaternions is denoted by $\mathrm{H}_{\alpha\beta}$ (Jafari, 2012). We express the basic operations in the $\vec{i}, \vec{j}, \vec{k}$ form. The addition becomes as

$$(a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) + (b_0 + b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$
$$= (a_0 + b_0) + (a_1 + b_1)\vec{i} + (a_2 + b_2)\vec{j} + (a_3 + b_3)\vec{k}$$

and the multiplication as

$$(a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k})(b_0 + b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$
$$= (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3)$$
$$+ (a_1b_0 + a_0b_1 - \beta a_3b_2 + \beta a_2b_3)\vec{i}$$
$$+ (a_2b_0 + \alpha a_3b_1 + a_0b_2 - \alpha a_1b_3)\vec{j}$$
$$+ (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)\vec{k}.$$

Given $q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, a_0 is called the *scalar part* of q, denoted by

$$S(q) = a_0,$$

and $a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ is called the *vector part* of q, denoted by
$\vec{V}(q)=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}.$

The *conjugate* of q is
$\overline{q}=a_0-a_1\vec{i}-a_2\vec{j}-a_3\vec{k}.$

The *norm* of q is
$N_q=\overline{q}q=q\overline{q}=a_0^2+\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2.$

The *inverse* of q with $N_q\neq 0,$ is
$q^{-1}=\frac{1}{N_q}\overline{q}.$

Clearly $qq^{-1}=1+0\vec{i}+0\vec{j}+0\vec{k}.$ Note also that $\overline{qp}=\overline{p}\overline{q}$ and $(qp)^{-1}=p^{-1}q^{-1}.$

De Moivre's Formula for Generalized Quaternions

We investigate the properties of the generalized quaternions in two different cases.

Case 1: α,β are positive numbers.

Let S_G^3 be the set of all unit dual generalized quaternions and S_G^2 be the set of all unit generalized vectors, that is,

$S_G^3=\{q\in H_{\alpha\beta}: N_q=1\}\subset H_{\alpha\beta},$

$S_G^2=\{\vec{V}(q)=a_1\vec{i}+a_2\vec{j}+a_3\vec{k} : N_{\vec{V}(q)}=\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2=1\}.$

Theorem 1. Under quaternionic multiplication, S_G^3 is a Lie group of dimension 3.

Proof: To show that S_G^3 with the multiplication is a group, let $f: H_{\alpha\beta}\to\mathbb{R}$ be a differentiable function given as
$f(q)=a_0^2+\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2.$

$S_G^3=f^{-1}(1)$ is a submanifold of $H_{\alpha\beta}$, since 1 is a regular value of function f. Also, the following maps $\mu: S_G^3\times S_G^3\to S_G^3$ sending (q,p) to qp and $\zeta: S_G^3\to S_G^3$ sending q to q^{-1} are both differentiable.

Every nonzero generalized quaternion $q=a_0+a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$ can be written in the polar form
$q=\sqrt{N_q}(\cos\theta+\vec{u}\sin\theta)$

where $\cos\theta=\frac{a_0}{\sqrt{N_q}}$ and $\sin\theta=\frac{\sqrt{\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2}}{\sqrt{N_q}}$. The unit generalized vector $\vec{u}$ is given by

$\vec{u}=(u_1,u_2,u_3)=\frac{1}{\sqrt{\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2}}(a_1,a_2,a_3),$

with $\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2\neq 0.$

For any $\vec{u}\in S_G^2$, since $\vec{u}^2=-1$ we have a natural generalization of Euler's formula for generalized quaternions

$$\begin{aligned} e^{\vec{u}\theta} &= 1+\vec{u}\theta-\frac{\theta^2}{2!}-\vec{u}\frac{\theta^3}{3!}+\frac{\theta^4}{4!}-\dots \\ &= 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\dots+\vec{u}(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\dots) \\ &= \cos\theta+\vec{u}\sin\theta, \end{aligned}$$

for any real number θ. For detailed information about Euler's formula, see (Whittlesey, 1990).

Lemma 1. For any $\vec{u}\in S_G^2$, we have

$$\left(\cos\theta_1+\vec{u}\sin\theta_1\right)\left(\cos\theta_2+\vec{u}\sin\theta_2\right)=\cos\left(\theta_1+\theta_2\right)+\vec{u}\sin\left(\theta_1+\theta_2\right).$$

Theorem 2. (De Moivre's formula) Let $q=e^{\vec{u}\theta}=\cos\theta+\vec{u}\sin\theta$ be a unit generalized quaternion. Then for any integer n;

$$q^n=e^{n\vec{u}\theta}=\cos n\theta+\vec{u}\sin n\theta.$$

Proof: The proof will be by induction on nonnegative integers $n.$ For $n=2$ and on using the validity of theorem as lemma 1, one can show

$$(\cos\theta+\vec{u}\sin\theta)^2=\cos 2\theta+\vec{u}\sin 2\theta$$

Suppose that $(\cos\theta+u\sin\theta)^n=\cos n\theta+\vec{u}\sin n\theta,$ we aim to show

$$(\cos\theta+u\sin\theta)^{n+1}=\cos(n+1)\theta+\vec{u}\sin(n+1)\theta.$$

Thus

$$\begin{aligned}(\cos\theta+\vec{u}\sin\theta)^{n+1}&=(\cos\theta+\vec{u}\sin\theta)^n(\cos\theta+\vec{u}\sin\theta)\\&=(\cos n\theta+\vec{u}\sin n\theta)(\cos\theta+\vec{u}\sin\theta)\\&=\cos(n\theta+\theta)+\vec{u}\sin(n\theta+\theta)\\&=\cos(n+1)\theta+\vec{u}\sin(n+1)\theta.\end{aligned}$$

The formula holds for all integers n;

$$q^{-1}=\cos\theta-\vec{u}\sin\theta,$$

$$\begin{aligned}q^{-n}&=\cos(-n\theta)+\vec{u}\sin(-n\theta)\\&=\cos n\theta-\vec{u}\sin n\theta.\end{aligned}$$

Special case: If $\alpha=\beta=1,$ then Theorem 2 holds for real quaternions, see (Cho,1998).

Example 1. Let $q=-\frac{1}{2}+\frac{1}{2}(\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}})=\cos\frac{2\pi}{3}+\frac{1}{\sqrt{3}}(\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}})\sin\frac{2\pi}{3}$ be a unit gen- eralized quaternion. Every power of this quaternion is found with the aid of theorem 3. For example, *9-th* and *53-th* powers are

$$\begin{aligned}q^9&=\cos 9\frac{2\pi}{3}+\frac{1}{\sqrt{3}}(\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}})\sin 9\frac{2\pi}{3}\\&=1,\end{aligned}$$

and

$$\begin{aligned}q^{53}&=\cos 53\frac{2\pi}{3}+\frac{1}{\sqrt{3}}(\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}})\sin 53\frac{2\pi}{3}\\&=-\frac{1}{2}-\frac{1}{2}(\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}}).\end{aligned}$$

Theorem 3. De Moivre’s formula implies that there are uncountably many unit generalized quaternions q satisfying $q^n=1$ for $n\geq 3$.

Proof: For every $\vec{u}\in S^2_G$, the unit generalized quaternion

$$q=\cos\frac{2\pi}{n}+\vec{u}\sin\frac{2\pi}{n}$$

is of order $n.$ For $n=1$ or $n=2,$ the generalized quaternion q is independent of $\vec{u}.$

Example 2. $q_1=\frac{1}{\sqrt{2}}+\frac{1}{2}(\frac{1}{\sqrt{\alpha}},0,\frac{1}{\sqrt{\alpha\beta}})=\cos\frac{\pi}{4}+\vec{u}\sin\frac{\pi}{4}$ is of order 8 and $q_2=\frac{1}{2}+\frac{1}{2}(\frac{1}{\sqrt{\alpha}},\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}})=\cos\frac{\pi}{3}+\vec{u}\sin\frac{\pi}{3}$ is of order 6.

Theorem 4. Let $q=\cos\theta+\vec{u}\sin\theta$ be a unit generalized quaternion. The equation $x^n=q$ has n roots, and they are

$$x_k = \cos(\frac{\theta + 2k\pi}{n}) + \vec{u}\sin(\frac{\theta + 2k\pi}{n}), \qquad k = 0,1,2,...,n-1.$$

Proof: We assume that $x = \cos\vartheta + \vec{u}\sin\vartheta$ is a root of the equation $x^n = q,$ since the vector parts of x and q are the same. From Theorem 3, we have

$x^n = \cos n\vartheta + \vec{u}\sin n\vartheta,$

thus, we find

$\cos n\vartheta = \cos\theta, \qquad \sin n\vartheta = \sin\theta,$

So, the n roots of q are $x_k = \cos(\frac{\theta + 2k\pi}{n}) + \vec{u}\sin(\frac{\theta + 2k\pi}{n}), \ k = 0,1,2,...,n-1.$

The relation between the powers of generalized quaternion can be found in the following theorem.

Theorem 5. Let q be a unit generalized quaternion with the polar form $q = \cos\theta + \vec{u}\sin\theta.$ If $m = \frac{2\pi}{\theta} \in \mathbb{Z}^+ - \{1\},$ then $q^n = q^p$ if and only if $n \equiv p(\mathrm{mod}\, m).$

Proof: Let $n \equiv p(\mathrm{mod}\, m).$ Then we have $n = am + p,$ where $a \in \mathbb{Z}.$

$$\begin{aligned} q^n &= \cos n\theta + \vec{u}\sin n\theta \\ &= \cos(am + p)\theta + \vec{u}\sin(am + p)\theta \\ &= \cos(a\frac{2\pi}{\theta} + p)\theta + \vec{u}\sin(a\frac{2\pi}{\theta} + p)\theta \\ &= \cos(p\theta + a2\pi) + \vec{u}\sin(p\theta + a2\pi) \\ &= \cos p\theta + \vec{u}\sin p\theta \\ &= q^p. \end{aligned}$$

Now suppose $q^n = \cos n\theta + \vec{u}\sin n\theta$ and $q^p = \cos p\theta + \vec{u}\sin p\theta.$ If $q^n = q^p$ then we get $\cos n\theta = \cos p\theta$ and $\sin n\theta = \sin p\theta,$ which means $n\theta = p\theta + 2\pi a, \ a \in \mathbb{Z}.$ Thus $n = p + \frac{2\pi}{\theta}a$ or $n \equiv p(\mathrm{mod}\, m).$

Example 3. Let $q_1 = \frac{1}{\sqrt{2}} + \frac{1}{2}(\frac{1}{\sqrt{\alpha}}, 0, \frac{1}{\sqrt{\alpha\beta}})$ be a unit generalized quaternion. From theorem 4, $m = \frac{2\pi}{\pi/4} = 8,$ so we have

$q = q^9 = q^{17} = ...$

$q^2 = q^{10} = q^{18} = ...$

$q^3 = q^{11} = q^{19} = ...$

$q^4 = q^{12} = q^{20} = ... = -1$

...

$q^8 = q^{16} = q^{24} = ... = 1.$

Case 2: Let α be a positive number and β be a negative number.

In this case, for a generalized quaternion $q = a_0 + a_1\vec{i} + a_2\vec{j} + a_3\vec{k},$ we can consider three different subcases.

Subcase (i): Let norm of generalized quaternion be positive and the norm of its vector part nega- tive, *i.e.*

$$N_q = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 > 0, \qquad \vec{V}_q = \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 < 0.$$

In this case, the polar form of q is defined as

$q = r(\cosh\vartheta + \vec{w}\sinh\vartheta),$

with

$r = \sqrt{N_q} = \sqrt{a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2},$

$\cosh\vartheta=\frac{a_0}{\sqrt{N_q}}$ and $\sinh\vartheta=\frac{\sqrt{-\alpha a_1^2-\beta a_2^2-\alpha\beta a_3^2}}{\sqrt{N_q}}$. The unit generalized vector $\vec{w}$ (axis of quarter- nion) is defined as

$$\vec{w}=(w_1,w_2,w_3)=\frac{1}{\sqrt{-\alpha a_1^2-\beta a_2^2-\alpha\beta a_3^2}}(a_1,a_2,a_3).$$

Special case: If $\alpha=1,\ \beta=-1,\ N_q=a_0^2+a_1^2-a_2^2-a_3^2>0$ and $\vec{V}_q=a_1^2-a_2^2-a_3^2<0.$ Then the q is a timelike quaternion with spacelike vector part and its polar form is

$$q=\sqrt{N_q}\,(\cosh\theta+\vec{\varepsilon}\sinh\theta),$$

where $\vec{\varepsilon}=\frac{1}{\sqrt{-a_1^2+a_2^2+a_3^2}}(a_1,a_2,a_3)$ is a spacelike unit vector in E_1^3 and $\vec{\varepsilon}*\vec{\varepsilon}=1.$ (Ozdemir, 2006)

Theorem 6. (De Moivre's formula) Let $q=r(\cosh\vartheta+\vec{w}\sinh\vartheta)$ be a generalized quaternion with $N_q>0,\ \vec{V}_q<0.$ Then for any integer $n,$ we have

$$q^n=r^n(\cosh n\vartheta+\vec{w}\sinh n\vartheta).$$

Proof: We use induction on positive integers $n.$ Assume that $q^n=r^n(\cosh n\vartheta+\vec{w}\sinh n\vartheta)$ holds. Then,

$$\begin{aligned}q^{n+1}&=r^n(\cosh n\vartheta+\vec{w}\sinh n\vartheta)\,r\,(\cosh\vartheta+\vec{w}\sinh\vartheta)\\&=r^{n+1}(\cosh n\vartheta+\vec{w}\sinh n\vartheta)(\cosh\vartheta+\vec{w}\sinh\vartheta)\\&=r^{n+1}[(\cosh n\vartheta\cosh\vartheta+\sinh n\vartheta\sinh\vartheta)+\vec{w}(\cosh n\vartheta\sinh\vartheta+\sinh n\vartheta\cosh\vartheta)\\&=r^{n+1}[\cosh(n+1)\vartheta+\vec{w}\sinh(n+1)\vartheta].\end{aligned}$$

Hence, the formula is ture.

Subcase (ii): Let the norm of generalized quaternion be positive and the norm of its vector part positive, *i.e.*

$$N_q=a_0^2+\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2>0,\qquad \vec{V}_q=\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2>0.$$

In the case, the polar form of q is defined as

$$q=r(\cos\theta+\vec{u}\sin\theta),$$

with

$$r=\sqrt{N_q}=\sqrt{a_0^2+\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2},$$

$\cos\theta=\frac{a_0}{\sqrt{N_q}}$ and $\sin\theta=\frac{\sqrt{\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2}}{\sqrt{N_q}}$. The unit generalized vector $\vec{u}$ (axis of quarter- nion) is defined as

$$\vec{u}=(u_1,u_2,u_3)=\frac{1}{\sqrt{\alpha a_1^2+\beta a_2^2+\alpha\beta a_3^2}}(a_1,a_2,a_3).$$

Special case: If $\alpha=1,\ \beta=-1,\ N_q=a_0^2+a_1^2-a_2^2-a_3^2>0$ and $\vec{V}_q=a_1^2-a_2^2-a_3^2>0.$ Then the q is a timelike quaternion with timelike vector part and its polar form is

$$q=\sqrt{N_q}\,(\cos\theta+\vec{u}\sin\theta),$$

where $\vec{u}$ is a timelike unit vector in E_1^3 and $\vec{u}*\vec{u}=-1.$ (Ozdemir, 2006)

Theorem 7. (De Moivre's formula) Let $q=r(\cos\theta+\vec{u}\sin\theta)$ be a generalized quaternion with $N_q>0,\ \vec{V}_q>0.$ Then

$$q^n=r^n(\cos n\theta+\vec{u}\sin n\theta),$$

for any integer $n,$

Proof: The proof of this theorem can be done using induction, similarly, to the proof of the Theorem 2.

Subcase (iii): The norm of generalized quaternion is negative, *i.e.*

$N_q = a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 < 0.$

Since $0 < a_0^2 < -\alpha a_1^2 - \beta a_2^2 - \alpha\beta a_3^2$ then $\alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2 < 0.$ In the case, the polar form of q is defined as

$q = r(\sinh\varphi + \vec{u}\cosh\varphi),$

with

$r = \sqrt{|N_q|} = \sqrt{|a_0^2 + \alpha a_1^2 + \beta a_2^2 + \alpha\beta a_3^2|},$

$\sinh\varphi = \frac{a_0}{\sqrt{|N_q|}}$ and $\cosh\varphi = \frac{\sqrt{-\alpha a_1^2 - \beta a_2^2 - \alpha\beta a_3^2}}{\sqrt{|N_q|}}.$ The unit generalized vector $\vec{u}$ (axis of quarternion) is defined as

$\vec{u} = (u_1, u_2, u_3) = \frac{1}{\sqrt{-\alpha a_1^2 - \beta a_2^2 - \alpha\beta a_3^2}}(a_1, a_2, a_3).$

Special case: If $\alpha = 1,\ \beta = -1$ and norm is negative number, *i.e.* $N_q = a_0^2 + a_1^2 - a_2^2 - a_3^2 < 0.$ Then the q is a spacelike quaternion and its polar form is

$q = \sqrt{|N_q|}(\sinh\varphi + \vec{u}\cosh\varphi).$

where $\vec{u}$ is a spacelike unit vector in E_1^3. The product of two spacelike quaternions is timelike. (Ozdemir, 2006)

Theorem 8. (De Moivre's formula) Let $q = r(\sinh\varphi + \vec{u}\cosh\varphi)$ be a generalized quaternion with $N_q < 0.$ Then for any integer $n,$ we have

$$q^n = \begin{cases} r^n(\sinh n\varphi + \vec{u}\cosh n\varphi), & n \ \text{is odd}. \\ r^n(\cosh n\varphi + \vec{u}\sinh n\varphi), & n \ \text{is even}. \end{cases}$$

Proof: The proof follows immediately from the induction.

CONCULSION

In this paper, we defined and gave some of algebraic properties of generalized quaternion and investigated the Euler's and De Moivre's formulas for these quaternions in several cases. The relation between the powers of generalized quaternions is given in theorem 5. We also showed that the equation $q^n = 1$ has uncountably many solutions for any general unit generalized quaternion (Theorem 4).

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